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On Acoustic Information for Motion

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Specific details of acoustic information that support the perception of moving sound sources are presented. The analyses demonstrate that higher order variables related to moving sound sources such as position, velocity, and time-to-arrival are observable from conjoint measurement of the time-varying acoustic variables of interaural-time-delay, Doppler shift, and average sound intensity. Motion structures the ambient acoustic array in such a way that the measured variables under consideration sufficiently specify motion kinematics when the system of forward equations, combined with their first-order differentials with respect to time, are inverted.

Over the last several decades, the ecological approach to perception has primarily been the province of vision researchers. Only recently has the auditory system received attention by proponents of this approach (Gaver, 1993). The acoustic medium is often viewed as a complementary source of information; after all, higher primates for the most part are highly visual beings. However, the physical characteristics of sound offer certain advantages in the detection of an object regardless of its location relative to the organism, such as the ability to detect the location and movement of a sound source in darkness or when the line of sight is occluded. Indeed, the survival of the animal may depend more on the initial detection of a sound source and whether it's moving, than on its initial visual appearance.

Much of the interest regarding an active observer in the environment has centered on the precise timing of interaction with an object relative to the observer or actor. Examples include catching a thrown ball, striking a ball with a bat, or avoiding a collision with a ball. Navigating through a maze of static objects involves similar abilities on the part of the active observer. These examples certainly seem natural for the visual sense, but what about audition? Although the physical nature

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of optics and acoustics are different, the information or meaning conveyed to the observer within certain constraints is assuredly the same.

Empirical evidence examining how dimensions of acoustic information contribute to the perception of what Gibson (1966) referred to as *higher order variables*, such as time-to-contact, velocity, trajectory, and range, are rare. Rosenblum, Carello, and Pastore (1987) examined the contributions of the Doppler frequency shift, sound intensity, and interaural-time-delay to judgments of impending collision. Observed patterns of judgments suggested that listeners were able to use the available acoustic information to determine time-to-contact. More recently, Schiff and Oldak (1990) showed that sighted observers use visual information more efficiently than acoustic information for judging time-to-contact. They found, however, that blind observers using only acoustic information are as accurate as sighted observers in judging impending collision, which would support the sufficiency of available information in both domains. Acoustic variables used by the bat for echolocation have been analyzed by Lee, van der Weel, Hitchcock, Matejowsky, and Pettigrew (1992) for information about time-to-contact. The active sonar world of echolocating bats is likely to be structured in an analogous fashion to that of the visual world. However, the bat is a highly specialized species whose primary source of sensory information is acoustic and is perhaps not the best model for human perception of motion.

The aim of this article is to lay out a mathematical framework for acoustic information that specifies higher order variables or the kinematics (changing spatial geometry) of a single moving sound source. Information is available to the degree that a unique mapping exists between some set of acoustic variables and a set of higher order variables. This consideration of information is generally in line with that of Gibson's (1966). Gibson viewed information as structure, and in the case presented here, kinematics is considered a form of structure evolving over time. It is certainly the case that not all possible acoustic variables are considered in this analysis. Therefore, should the information analysis fail to produce an inverse solution, it is still possible that a solution exists for some other set of acoustic variables. On the other hand, should the analysis yield an inverse solution, then the selected set of acoustic variables are, in principle, sufficient to specify the considered higher order variables. Under conditions in which more than one solution exists, the proposed information analysis illuminates ways in which disambiguating constraints could be brought to bear on the set of solutions. Although exact mappings are often complex, they serve to illustrate the precise algebraic fusion of the acoustic variables that specify the higher order variables.

ECOLOGICAL ACOUSTICS

Ecological acoustics is the physics of sound relevant to the animal in its environment. Koenderink (1994) defined *ecological physics* as "the science of the causal nexus that

nexus that closes the cyclical process of environmentally guided behavior of some agent through that agent's world." Taking an ecological approach to acoustics entails an analysis of the physical mechanics of sound sources and the propagation of sound with respect to a dynamic observer.

In the following sections, several dimensions of ecological acoustics are detailed, beginning with a brief introduction to several dimensions of physical acoustics. The following discussion is not intended to be an exhaustive coverage of ecological acoustics, but rather it is intended to convey a general approach to informational analysis specific to the acoustic medium. For purposes here, only sound sources located on the horizontal plane are considered. It is acknowledged that this analysis is incomplete; however, consideration of full 3-D auditory space involves an account of complex acoustic filtering induced by the head and pinnae (external ears), which are idiosyncratic to the observer. These characteristics are commonly described mathematically by so-called head-related transfer functions. A review of this source of spatial information can be found in Wightman and Jenison (1995). Restricting the analysis to the horizontal plane affords a tractable first step in an analysis of information, and this constraint is reasonable given that ecological motion is often restricted to the horizontal plane. However, the goal is to ultimately provide an information analysis of full 3-D auditory space.

Physical Acoustics

Waves are a common phenomenon in nature and an important means by which information is conveyed from a physical event to the observer. The concept of a point source generating waves governed by linear acoustic equations is a convenient model of sound radiation. Any small source with a time-varying mass of fluid (air) has all the attributes of a point source provided the dimensions of the source are small relative to the wavelength, and the sound field of interest is a geometric far-field. The radiating geometry of the simple point source can be characterized by the type of source—monopole, dipole, and multipole. The monopole is omnidirectional, that is, the same in any direction equidistant from the source, whereas the multipole has more complex patterns that are direction dependent.

Sound Intensity

The intensity of sound will decay as the pressure wave travels away from the source due to several factors. Temperature gradients caused by compression and rarefaction of the propagating wave will naturally result in energy loss. Sound absorption in air is wavelength dependent and specified by the attenuation coefficient α. Sometimes the term attenuation length, α^{-1}, is used to describe the distance over

which the intensity decreases to e^{-1}, in which e is the base of the natural logarithm. Intensity also diminishes with distance because the rate of energy change or power is spread out over the surface area of a spreading sphere, hence inversely proportional to the square of the distance (r) to the receiver. Considering both geometrical spreading and absorption, intensity as a function of distance can be written

$$I_{avg} = \frac{\eta}{r^2} e^{-\alpha r} \tag{1}$$

in which r is the distance from the sound source, η is proportional to the intrinsic radiating power, and α is the attenuation coefficient. At short ranges from the source intensity is dominated by spherical spreading, whereas distances well beyond the attenuation length lead to more rapid decreases in intensity.

Interaural-Time-Delay

Perhaps the most striking dissimilarity between auditory perception and visual perception is the manner in which the ambient energy array projects onto the respective sensory transducers. For vision, a 3-D projection onto a planar surface with respect to the line of sight is generally assumed as a first-order approximation. For audition, the 3-D acoustic environment projects to only two points on opposing positions on the head. Projection angles must be considered in terms of acoustic path delays or more complicated spectral differences. The geometry of a spherical wave, becoming planar when the source is far away, striking a spherical head is shown in Figure 1. An acoustic delay between the left and right ears (considered here to be point receivers), denoted by δ, occurs, which is a function of the incident angle θ (azimuthal direction) with respect to the median line on the horizontal plane. By initially making the simplifying assumption of a transparent acoustic path, the acoustic delay can be shown to be

$$\delta = \frac{2a}{c} \sin\theta \tag{2}$$

in which a is the radius of an ideal spherical head and c is the velocity of sound. This function is frequency independent for source frequencies greater than 2 kHz. At lower frequencies, a spherical path for the shadowed ear (observation point) must be considered (see Woodworth & Schlosberg, 1962) in which

$$\delta = \frac{a}{c}(\theta + \sin\theta). \tag{3}$$

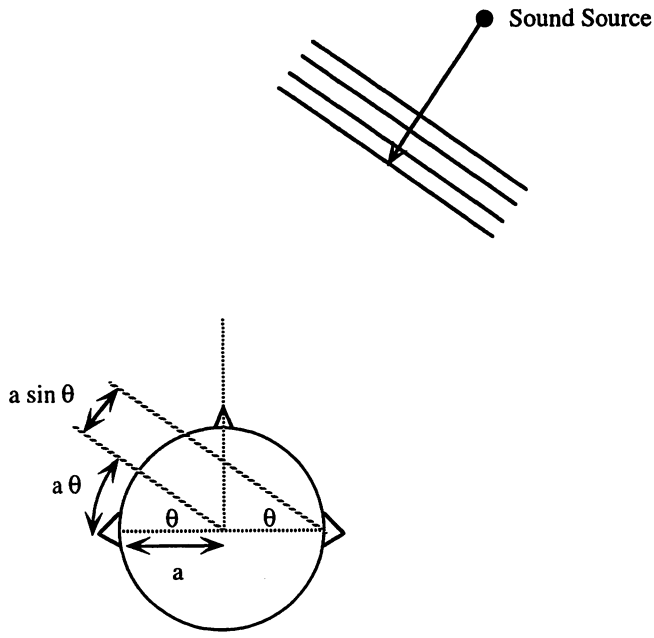

FIGURE 1 Geometric relation between a point sound source and the left and right points on the sphere. θ is the angle of incidence perpendicular to the planar wavefront when the source is of sufficient range *r*.

Kuhn (1977) showed that Equation 3 can be approximated from analysis of acoustic pressures on the surface of a rigid sphere by the expression

$$\delta = \frac{3a}{c}\sin\theta \tag{4}$$

which is also frequency independent at frequencies below 500 Hz, in which interaural-time-delay is a reliable determiner of sound source direction. It is then possible to determine a value of δ, as long as the range of frequencies of the sound source is known.

Doppler Shifted Frequency

The Doppler effect corresponds to a frequency shift associated with a sound source moving through a homogeneous medium. Pressure wave crests emerge from the sound source at intervals corresponding to the acoustic wavelength. Each crest spreads spherically out from the point of origin at the speed of sound *c*. The

successively generated spheres of wave crests are closer together ahead of the sound source but farther apart behind the source. For a stationary observer, the measured frequency corresponds to the number of crests per unit time, so the composite frequencies will be higher when the observer is in front of the moving sound source and less when behind the moving sound source. A familiar example is the shift in frequency of an ambulance siren as the vehicle approaches, passes, and then recedes. The well-known lawful dependence of the Doppler-shifted frequency, here denoted ψt, on velocity of the sound source relative to an observer is

$$\psi t = f\left(1+\frac{v}{c}\cos\varphi t\right) \tag{5}$$

in which f is the intrinsic frequency of the sound source, v is the velocity magnitude (speed) and c is the speed of sound. The shifted frequency ψ_t depends only on the velocity component directed toward the observer with angle φ_t (see Figure 2). This result holds true regardless of the time history of the trajectory.

OBSERVABILITY OF HIGHER ORDER VARIABLES

Inverse solutions can serve to demonstrate the sufficiency of acoustic variables in specifying higher order variables of motion. Higher order variables refer to the various states of a moving sound source: position, velocity, time-to-arrival, and so

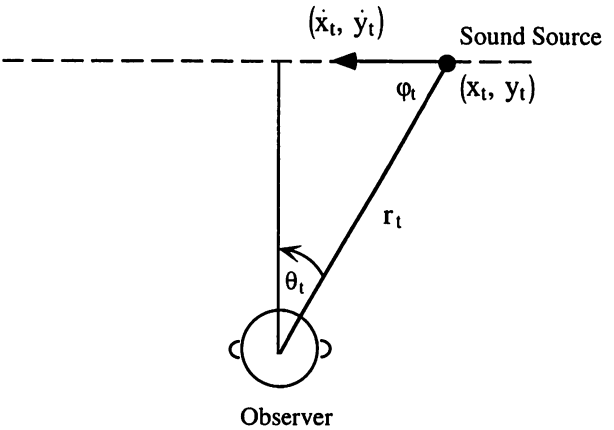

FIGURE 2 Kinematics of rectilinear motion of a sound source relative to an observer.

forth. Algorithmic implementation of a derived formula in a particular sensory system is not the goal of this analysis, but more fundamentally whether unique (or nearly so) solutions can be shown to exist. The mathematical technique of showing the existence of an inverse solution is developed from the nonlinear systems literature on deterministic[1] observability (Griffith & Kumar, 1971; Hermann & Krener, 1977; Hwang & Seinfeld, 1972; Kou, Elliott, & Tarn, 1973; Mohler & Hwang, 1988). The fundamental question addressed by analysis of observability is whether the state of a dynamical process or system (kinematics) can be uniquely determined from its output data at a given point in time. The process is formally defined as a function of the state vector $\vec{s}_t$, such that $\vec{m}_t = F(\vec{s}_t)$, in which $F(\vec{s}_t)$ represents the set of forward equations. The physical process is observable at some time t if the measurement vector $\vec{m}_t$ is sufficient to determine $\vec{s}_t$ uniquely. The strategy used in the observability literature is one of differentiating $F(\vec{s}_t)$ up to a necessary order, such that sufficient equations exist to solve for $\vec{s}_t$.

System of Forward Equations

The construction of forward equations first entails transformation of the physical equations (Equations 1, 2, 4, and 5) from polar to Cartesian coordinates. The natural log transform of intensity was used to mathematically approximate the perceptual dimension of loudness. Coordinate transformation is motivated by the fact that Cartesian coordinates are mathematically easier to work with in this case, involving radicals rather than trigonometric functions. There is no gain or loss of information as a result of these transformations. Several simplifying assumptions were made. First, the sound source was assumed to be monopolar and traveling well within the range of the attenuation length, such that intensity is dominated by spherical spreading. Second, the constants accounting for the acoustic delay in the forward equation for interaural-time-delay δ were dropped, such that δ ranges between -1 and 1. It is reasonable to assume that these constants could be known to the observer. Third, it was assumed that velocity of the sound source remains stationary over a short span of time. A more general system of equations could include acceleration.

The initial set of three forward equations are defined using six higher order state variables: position (x, y), velocity $(\dot{x}, \dot{y})$, intrinsic frequency (f), and intrinsic power (η). Although the intrinsic frequency and power are not kinematic variables, they are unknown and therefore considered as higher order variables that define sound source attributes. A system of three equations is not sufficient to solve for the specified set of unknown variables. The strategy then becomes one of differentiating the set of forward equations with respect to time until sufficient equations are

[1] *Deterministic* refers to systems evaluated without noisy measurements, in contrast to stochastic observability evaluated under uncertainty.

available to algebraically solve for the inverse. It is notable that differentiation of natural log intensity w.r.t. time eliminates the dependence on acoustic power of the resulting differential. The full system of six equations following first-order differentiation are shown in Table 1.

Several invariant characteristics are worth mentioning for both the forward and inverse equations. First, δ_t^2 will always fall in the range between 0 and 1. Second, the Doppler shift is always in a downward direction, regardless of whether the sound source is approaching or receding; consequently, $\dot{\psi}_t$ is always negative, as long as the velocity remains locally stationary.

Inverse Solutions

Table 2 lists the algebraic inverse solutions for the higher order variables. Once a sufficient system of forward equations is established, the process of algebraic elimination is relatively straightforward. The strategy for solving the unknown state variables is to first transform the equations into normal polynomial form (see Appendix A). Then, the method of Gröbner bases is used to transform the original system of polynomials into a well-structured system of polynomials that has the same set of roots (solutions) as the original system. An example of a well-structured system of polynomials is one in which the system of polynomials is brought into triangular form in which the last polynomial is univariate, or better yet, each polynomial in the system is univariate for each unknown state variable. General methods for constructing a set of Gröbner bases can be found in Adams and Loustaunau (1994), and the algorithm is available in most symbolic algebra packages (Maple, Mathematica, and Derive). As a check to ensure the validity of the inverse equations, the resulting solutions are substituted back into the forward equations and evaluated.

Position and Velocity

Unique inverse solutions exist for x_t and $\dot{x}_t$; however, two solutions exist for y_t and $\dot{y}_t$ denoted by the binary sign . This mathematical result is verified in the auditory

TABLE 1
Six Equations Showing First-Order Differentiation

Interaural-Time-Delay

$$\delta_t = \frac{x_t}{\sqrt{x_t^2 + y_t^2}} \quad (6)$$

$$\dot{\delta}_t = \frac{y_t(\dot{x}y_t - \dot{y}x_t)}{\sqrt{x_t^2 + y_t^2}^3} \quad (7)$$

Intensity

$$I_t = \ln\left(\frac{\eta}{x_t^2 + y_t^2}\right) \quad (8)$$

$$\dot{I}_t = \frac{-2(\dot{x}x_t + \dot{y}y_t)}{x_t^2 + y_t^2} \quad (9)$$

Doppler Shift

$$\psi_t = \acute{f}\left(1 - \frac{\dot{x}x_t + \dot{y}y_t}{c\sqrt{x_t^2 + y_t^2}}\right) \quad (10)$$

$$\dot{\psi}_t = -f\,\frac{(\dot{x}y_t - \dot{y}x_t)^2}{c\sqrt{x_t^2 + y_t^2}^3} \quad (11)$$

TABLE 2
Algebraic Inverse Solutions for Higher Order Variables

Position

$$x_t = 2\frac{c\,\delta_t\,\dot{\psi}_t\left(\delta_t^2 - 1\right)}{\Phi_t} \quad (12)$$

$$y_t = \pm 2\frac{c\,\dot{\psi}_t\sqrt{1-\delta_t^2}\,\left(\delta_t^2 - 1\right)}{\Phi_t} \quad (13)$$

Time-to-arrival

$$\tau_x = \frac{x_t}{\dot{x}_t} = \frac{2\delta_t}{2\dot{\delta}_t - \dot{I}_t\delta_t} \quad (19)$$

$$\tau_y = \frac{y_t}{\dot{y}_t} = \frac{2\left(\delta_t^2 - 1\right)}{2\delta_t\dot{\delta}_t + \dot{I}_t\left(1-\delta_t^2\right)} \quad (20)$$

Velocity

$$\dot{x}_t = \frac{c\,\dot{\psi}_t\left(\delta_t^2 - 1\right)\left(2\dot{\delta}_t - \dot{I}_t\delta_t\right)}{\Phi_t} \quad (14)$$

$$\dot{y}_t = \pm\frac{c\,\dot{\psi}_t\sqrt{1-\delta_t^2}\,\left(2\,\delta_t\dot{\delta}_t + \dot{I}_t\left(1-\delta_t^2\right)\right)}{\Phi_t} \quad (15)$$

Range

$$r_t = \sqrt{x_t^2 + y_t^2} = 2\frac{c\,\dot{\psi}_t\left(\delta_t^2 - 1\right)}{\Phi_t} \quad (21)$$

Intrinsic Frequency and Power

$$f_t = \frac{\Phi_t}{2\dot{\delta}_t^2} \quad (16)$$

$$\eta_t = \left[2\frac{c\,\dot{\psi}_t\left(\delta_t^2 - 1\right)}{\Phi_t}\right]^2 e^{I_t} \quad (17)$$

Where

$$\Phi_t = \left(1 - \delta_t^2\right)\dot{\psi}_t\dot{I}_t + 2\,\psi_t\dot{\delta}_t^2 \quad (18)$$

psychophysics literature in the form of now well-known "front–back" localization confusions (Wightman & Kistler, 1993). In 3-D space, this ambiguity would exist not as two solutions on the horizontal plane, but as an edge of a cone concentric about the aural axis, commonly referred to as the cone-of-confusion, unless other constraints are brought to bear on the inverse. In the strictest sense, x_t and $\dot{x}_t$ would be considered observable from acoustic variables; y_t and $\dot{y}_t$ would not be considered observable. Although ambiguity exists for the y dimension, it is restricted to a binary

decision on the horizontal plane. As will be discussed later, this ambiguity can be removed via active maneuvering by the observer, as well as by constraints imposed by acoustic filtering of the head and pinnae. Also, the signs of the equations for y_t and $\dot{y}_t$ are coupled, such that the ratio of the two variables cancels the ambiguity.

Time-to-Arrival

Time-to-contact is a useful quantity for controlling actions that involve the interception of, and perhaps more importantly, the avoidance of a moving object. Time-to-contact, specified by the variable *tau* (τ), has been defined in the vision literature by David Lee and his colleagues (1976; Lee & Young, 1985) as the distance from any point of contact divided by the collinear velocity of the moving object. Lee showed that the inverse of the relative rate of dilation of an optical angle Ω defines τ for a direct rectilinear approach and constant velocity, such that $\tau = 2\frac{\Omega}{\dot{\Omega}}$. Shaw, McGowan, and Turvey (1991) analyzed the acoustic near-field intensity produced by collinear relative movement between a sound source and an observer and showed τ to be related to the inverse of the relative change in average intensity, and Tresilian (1990, 1991) advanced a general time-to-interception to an arbitrary angle with respect to the observer.

In this analysis, time-to-arrival is framed in Cartesian coordinates and derived by taking the ratios of the position coordinates relative to their respective velocities, $\tau_x = \frac{x_t}{\dot{x}_t}$ and $\tau_y = \frac{y_t}{\dot{y}_t}$. Here, the tau equations are referenced to arrivals at the x and y axes respectively, with the observer located at the origin. Clearly, a direct collision results if $\dot{\delta}_t = 0$, which reduces to the equality of τ_x and τ_y. It is interesting to note that the contribution of Doppler cancels out, and interaural-time-delay and intensity change suffice. The resulting equations for tau differ from previous derivations only in the use of log intensity, rather than intensity magnitude. Here, the equations for time-to-arrival are simply an extension of the general analysis of observability. Both τ_x and τ_y are observable higher order variables, given the coupled sign for y_t and $\dot{y}_t$, as mentioned earlier.

Source Variables: Intrinsic Frequency and Power

Intrinsic frequency, denoted f_t in Table 2, refers to the frequency of the sound source that would be observed in the case in which the source and observer are stationary relative to one another. When the sound source is moving relative to the observer, there actually is an instant in time at which the intrinsic frequency can be directly

measured by the observer. This occurs when φ_t is equal to $\pm 90°$ (see Equation 5 and Figure 2), which is commonly referred to as the point of closest approach. However, the intrinsic frequency is technically observable at all other times as shown by Equation 16 in Table 2. The source acoustic power, η_t, is also technically observable (Equation 17). Although these two source variables are not kinematic variables, the fact that unique inverse solutions exist for both have interesting implications. Under static conditions, the intrinsic frequency could still be measured by the observer; however, acoustic power would be ambiguous, unless the range of the sound source was known. Motion, in effect, induces sufficient constraints to observe acoustic power.

CONCLUSIONS AND FUTURE DIRECTIONS

These analyses show that interaural-time-delay, intensity, and Doppler can sufficiently specify motion kinematics restricted to the horizontal plane for the higher order variables, with the exception of those related to the y dimension, which have binary solutions. Although this ambiguity would be problematic for a static observer, an active observer can interrogate the ambient acoustic array (e.g., via head turning), thus constraining the binary solutions for y_t and $\dot{y}_t$ to a single solution. There are several methods for formalizing an active maneuver, the most straightforward being the introduction of an acceleration state variable. The introduction of additional state variables would necessitate second-order differentiation of the forward equations with respect to time.

So are these the only acoustic sources of information for motion? Certainly not. As mentioned earlier, complex changes in the spectral pattern of the sound wave, due to the shape of the observer's head and pinnae, occur as a sound source moves relative to the observer. We could extend the analysis of observability as long as continuous forward equations characterizing the spectral pattern dependence on higher order state variables can be generated. One approach to extending the system of forward equations to full 3-D space would be to use the following spherical-function approximation to the directional dependent head-related transfer functions, which include dependence on both azimuth (θ) and elevation (ϕ):

$$H_\omega(\theta,\phi) = \sum_{j=1}^{J} w_{\omega j} e^{k_j[\sin\phi \sin\beta_j \cos(\theta-\alpha)+\cos\phi\cos\beta_j]} \tag{22}$$

The index ω represents samples of spectral magnitude $H_\omega(\theta, \phi)$, and the remaining variables (w, κ, α, β) are all estimated parameters used in the functional approximation to an individual's head-related transfer functions (see Jenison & Fissell, 1996, for details). Although this function adds an additional level of complexity to the analysis of observability, it affords the necessary elevational dependency for 3-D

space. Transforming from polar to Cartesian coordinates and differentiating Equation 22 would then generate the state variables of position z_t and velocity $\dot{z}_t$ (in addition to x_t, $\dot{x}_t$, y_t, and $\dot{y}_t$). The forward equations in Table 1 would also need to be extended to include z_t and $\dot{z}_t$. The full 3-D system certainly represents the logical, although technically challenging, next step in acoustic information analysis of motion. However, whether or not elevational information is included in the analysis, the fact remains that achieving a solution for an inverse implies only a sufficient set of conditions for observability, but does not imply that this is the only sufficient system of forward equations.

Given the availability of motion information in the acoustic environment, the question then arises as to whether or not it is actually used by the organism. Neural circuits capable of detecting this information may develop through self-organizing processes or cross-modal modulation by the visual system. Knudsen and Brainard (1991) found that the alignment of sensory maps are controlled during ontogeny by visual instruction of the auditory spatial tuning of neurons in the optic tectum of developing barn owls. Given a plastic neural architecture with the capacity to learn, detectors of higher order dimensions important to the survival of the animal may well form commensurate with available information rather than with measured dimensions of the physical stimulus. Gibson's (1966) notion of a perceptual system that overlaps sensory modalities and, to use Gibson's term, "resonates" to stimulus information rather than physical images, is in general agreement with emerging physiological evidence.

The forward equations derived in this article also serve as a basis for waveform synthesis that afford controlled tests of human participants' capacity to use auditory information to discriminate the higher order variables of motion, such as time-to-arrival, velocity, and intrinsic frequency. Although the theoretical analyses show that sufficient information is available to the observer with respect to these higher order variables, it remains to be known whether the human observer has the capacity to "pick up" this information, particularly under conditions of uncertainty. Currently, we are pursuing psychoacoustic studies employing paradigms very similar to those used in visual studies of time-to-arrival, for example, Todd (1981) and Regan and Hamstra (1993). Approaching virtual-sounds presented under headphones replace approaching visual objects presented on a computer monitor, and participants are asked to make time-to-arrival discriminations. The functions in this article have been useful in constructing statistically optimal estimators that are framed as Kalman filters, which provide benchmarks for ideal psychophysical performance of acoustic time-to-arrival discrimination (Jenison, 1996). In practice, Kalman filters serve to extend tests of deterministic observability, described in this article, to that of the more useful theoretical framework of stochastic observability for dynamical-acoustic systems (Kalman & Bucy, 1961). Psychophysical studies of intrinsic-frequency discrimination of moving sounds are also in progress (Neelon & Jenison, in press). The theoretical basis for these studies is the fact that intrinsic

frequencies of a sound source under motion are never actually measured by the observer because they are Doppler shifted, but intrinsic frequencies are observable, in principle, as shown by Equation 16 in Table 2. Again, the empirical question is whether or not human observers are capable of using the available information to discriminate or estimate intrinsic frequencies of moving sounds.

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APPENDIX A

Derivation of Inverse Equations

The following system of nonlinear equations is taken from Table 1 in the text. The expressions are generated by Maple V Release 4, in which the $\frac{\partial}{\partial t}$ operator replaces the "dot" notation used in the text.

Derivatives are treated just like any other variable in the algebraic system of equations. The original system of forward equations taken from Table 1 is organized as follows, with the addition of the expression for range r_t, which serves to remove the radicals (square roots) from the system of equations. Intrinsic power is solved separately.

$$\frac{x_t}{r_t} = \delta_t \tag{A1}$$

$$\frac{y_t\left(-x_t\left(\frac{\partial}{\partial t}y_t\right)+\left(\frac{\partial}{\partial t}x_t\right)y_t\right)}{r_t^3} = \frac{\partial}{\partial t}\delta_t \tag{A2}$$

$$f\left(1-\frac{\left(\frac{\partial}{\partial t}x_t\right)x_t + y_t\left(\frac{\partial}{\partial t}y_t\right)}{r_t c}\right) = \psi_t \tag{A3}$$

$$-\frac{f\left(-x_t\left(\frac{\partial}{\partial t}y_t\right)+\left(\frac{\partial}{\partial t}x_t\right)y_t\right)^2}{r_t^3 c} = \frac{\partial}{\partial t}\psi_t \tag{A4}$$

$$-2\frac{\left(\frac{\partial}{\partial t}x_t\right)x_t + x_t\left(\frac{\partial}{\partial t}y_t\right)}{r_t^2} = \frac{\partial}{\partial t}I_t \tag{A5}$$

$$x_t^2 + y_t^2 = r_t^2 \tag{A6}$$

First, the right-hand side of the equations are moved to the left-hand side to bring them into polynomial form.

$$\frac{x_t}{r_t} - \delta_t = 0 \tag{A7}$$

$$\frac{y_t\left(-x_t\left(\frac{\partial}{\partial t}y_t\right)+\left(\frac{\partial}{\partial t}x_t\right)y_t\right)}{r_t^3} - \left(\frac{\partial}{\partial t}\delta_t\right) = 0 \tag{A8}$$

$$f\left(1 - \frac{\left(\frac{\partial}{\partial t}x_t\right)x_t + y_t\left(\frac{\partial}{\partial t}y_t\right)}{r_t c}\right) - \psi_t = 0 \tag{A9}$$

$$-\frac{f\left(-x_t\left(\frac{\partial}{\partial t}y_t\right)+\left(\frac{\partial}{\partial t}x_t\right)y_t\right)^2}{r_t^3 c} - \left(\frac{\partial}{\partial t}\psi_t\right) = 0 \tag{A10}$$

$$-2\frac{\left(\frac{\partial}{\partial t}x_t\right)x_t + x_t\left(\frac{\partial}{\partial t}y_t\right)}{r_t^2} - \left(\frac{\partial}{\partial t}I_t\right) = 0 \tag{A11}$$

$$x_t^2 + y_t^2 - r_t^2 = 0 \tag{A12}$$

The system of rational expressions is normalized.

$$-\frac{-x_t + \delta_t r_t}{r_t} = 0 \tag{A13}$$

$$-\frac{y_t x_t\left(\frac{\partial}{\partial t}y_t\right) - \left(\frac{\partial}{\partial t}x_t\right)y_t^2 + \left(\frac{\partial}{\partial t}\delta_t\right)r_t^3}{r_t^3} = 0 \tag{A14}$$

$$\frac{fr_t c - f\left(\frac{\partial}{\partial t}x_t\right)x_t - fy_t\left(\frac{\partial}{\partial t}y_t\right) - \psi_t r_t c}{r_t c} = 0 \qquad \text{A15}$$

$$-\frac{fx_t^2\left(\frac{\partial}{\partial t}y_t\right)^2 - 2fx_t\left(\frac{\partial}{\partial t}y_t\right)\left(\frac{\partial}{\partial t}x_t\right)y_t + f\left(\frac{\partial}{\partial t}x_t\right)^2 y_t^2 + \left(\frac{\partial}{\partial t}\psi_t\right)r_t^3 c}{r_t^3 c} = 0 \qquad \text{A16}$$

$$-\frac{2\left(\frac{\partial}{\partial t}x_t\right)x_t + 2x_t\left(\frac{\partial}{\partial t}y_t\right) + \left(\frac{\partial}{\partial t}l_t\right)r_t^2}{r_t^2} = 0 \qquad \text{A17}$$

$$x_t^2 + y_t^2 - r_t^2 = 0 \qquad \text{A18}$$

Rational expressions are reduced to final polynomial form by eliminating the denominator. Note that each polynomial is multivariate with respect to the unknown state variables, thus rendering the system of polynomials difficult to solve.

$$x_t - \delta_t r_t = 0 \qquad \text{A19}$$

$$-y_t x_t\left(\frac{\partial}{\partial t}y_t\right) + \left(\frac{\partial}{\partial t}x_t\right)y_t^2 - \left(\frac{\partial}{\partial t}\delta_t\right)r_t^3 = 0 \qquad \text{A20}$$

$$fr_t c - f\left(\frac{\partial}{\partial t}x_t\right)x_t - fy_t\left(\frac{\partial}{\partial t}y_t\right) - \psi_t r_t c = 0 \qquad \text{A21}$$

$$-fx_t^2\left(\frac{\partial}{\partial t}y_t\right)^2 + 2fx_t\left(\frac{\partial}{\partial t}y_t\right)\left(\frac{\partial}{\partial t}x_t\right)y_t - f\left(\frac{\partial}{\partial t}x_t\right)^2 y_t^2 - \left(\frac{\partial}{\partial t}\psi_t\right)r_t^3 c = 0 \qquad \text{A22}$$

$$-2\left(\frac{\partial}{\partial t}x_t\right)x_t - 2x_t\left(\frac{\partial}{\partial t}y_t\right) - \left(\frac{\partial}{\partial t}l_t\right)r_t^2 = 0 \qquad \text{A23}$$

$$x_t^2 + y_t^2 - r_t^2 = 0 \qquad \text{A24}$$

The method of Gröbner bases is a technique for eliminating variables from a multivariate polynomial system of equations and is available in Maple. During the elimination process, new polynomials are formed from pairs of old ones. The order in which unknowns are eliminated is specified by an ordered list of terms used by the Maple function *gsolve*. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables have been successfully eliminated and separated as far as possible. The Maple commands to obtain the Gröbner basis set are as follows:

```
> with (grobner);
> newsys := gsolve(oldsys, [f,x,y,xdot,ydot,r]);
```

in which *oldsys* corresponds to the original system of polynomials (Equations A19–A24) and *newsys* corresponds to the reduced polynomials (Equations A25–A30) having the same roots as the original system. The ordered list [f,x,y,xdot,ydot,r] corresponds to the specified order in which the unknowns are eliminated. This order was chosen as the most efficient; however, any permutation of the variable list will yield this set of polynomials. The unknown variables are highlighted in gray.

$$-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right) + 2\left(\frac{\partial}{\partial t}\delta_t\right)^2 f = 0 \qquad \text{A25}$$

$$2c\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^3 - 2\delta_t c\left(\frac{\partial}{\partial t}\psi_t\right) + \left(-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\right)x_t = 0 \qquad \text{A26}$$

$$(2\delta_t^2 - 2)\left(\frac{\partial}{\partial t}y_t\right) + \left(-\left(\frac{\partial}{\partial t}I_t\right) - 2\delta_t\left(\frac{\partial}{\partial t}\delta_t\right) + \delta_t^2\left(\frac{\partial}{\partial t}I_t\right)\right)y_t = 0 \qquad \text{A27}$$

$$2c\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,2}\left(\frac{\partial}{\partial t}\delta_t\right)-c\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,3}\left(\frac{\partial}{\partial t}I_t\right)- \qquad \text{A28}$$
$$2c\left(\frac{\partial}{\partial t}\psi_t\right)\left(\frac{\partial}{\partial t}\delta_t\right)+c\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t\left(\frac{\partial}{\partial t}I_t\right)+$$
$$\left(-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2+\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,2}-\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\right)\left(\frac{\partial}{\partial t}x_t\right)=0$$

$$c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,6}\left(\frac{\partial}{\partial t}I_t\right)^2-3c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,4}\left(\frac{\partial}{\partial t}I_t\right)^2- \qquad \text{A29}$$
$$4c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,5}\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\delta_t\right)+8c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,3}\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\delta_t\right)+$$
$$3c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,2}\left(\frac{\partial}{\partial t}I_t\right)^2+4c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,4}\left(\frac{\partial}{\partial t}\delta_t\right)^2-$$
$$4c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,2}\left(\frac{\partial}{\partial t}\delta_t\right)^2-4c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\left(\frac{\partial}{\partial t}I_t\right)\delta_t\left(\frac{\partial}{\partial t}\delta_t\right)-$$
$$c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\left(\frac{\partial}{\partial t}I_t\right)^2+$$
$$\left(\begin{array}{l}4\psi_t^{\,2}\left(\frac{\partial}{\partial t}\delta_t\right)^4-4\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,2}+\\ 4\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)+\left(\frac{\partial}{\partial t}I_t\right)^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,4}-\\ 2\left(\frac{\partial}{\partial t}I_t\right)^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\delta_t^{\,2}+\left(\frac{\partial}{\partial t}I_t\right)^2\left(\frac{\partial}{\partial t}\psi_t\right)^2\end{array}\right)\left(\frac{\partial}{\partial t}y_t\right)^2=0$$

$$2c\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,2}-2c\left(\frac{\partial}{\partial t}\psi_t\right)+ \qquad \text{A30}$$
$$\left(-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2+\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^{\,2}-\left(\frac{\partial}{\partial t}I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\right)\ r_t=0$$

The unknown variables are now easily solved over the new set of Gröbner bases, because the unknowns are univariate for each polynomial, with the exception of Equation A27, which is bivariate w.r.t. y_t and y_t. Equation A29 is univariate w.r.t.

$\frac{\partial}{\partial t} y_t$; consequently, y_t was solved first in terms of $\frac{\partial}{\partial t} y_t$, and then the solution for $\frac{\partial}{\partial t} y_t$ was backsubstituted.

$$f = -\tfrac{1}{2} \frac{-2\psi_t \left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)}{\left(\frac{\partial}{\partial t}\delta_t\right)^2} \qquad \text{A31}$$

$$x_t = -2 \frac{\delta_t c\left(\frac{\partial}{\partial t}\psi_t\right)\left(\delta_t^2 - 1\right)}{-2\psi_t \left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \qquad \text{A32}$$

$$\frac{\partial}{\partial t} x_t = \frac{c\left(\frac{\partial}{\partial t}\psi_t\right)\left(-2\delta_t^2\left(\frac{\partial}{\partial t}\delta_t\right) + \delta_t^3\left(\frac{\partial}{\partial t} I_t\right) + 2\left(\frac{\partial}{\partial t}\delta_t\right) - \delta_t\left(\frac{\partial}{\partial t} I_t\right)\right)}{-2\psi_t \left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \qquad \text{A33}$$

$$y_t = -2 \frac{\left(\delta_t^2 - 1\right)\left(\frac{\partial}{\partial t} y_t\right)}{-\left(\frac{\partial}{\partial t} I_t\right) - 2\delta_t\left(\frac{\partial}{\partial t}\delta_t\right) + \delta_t^2\left(\frac{\partial}{\partial t} I_t\right)} \qquad \text{A34}$$

Equation A29 is quadratic; consequently, there are two solutions for $\frac{\partial}{\partial t} y_t$. The only difference between the two solutions is the sign.

$$\frac{\partial}{\partial t} y_t = \frac{\sqrt{1-\delta_t^2}\left(-\left(\frac{\partial}{\partial t} I_t\right) - 2\delta_t\left(\frac{\partial}{\partial t}\delta_t\right) + \delta_t^2\left(\frac{\partial}{\partial t} I_t\right)\right)c\left(\frac{\partial}{\partial t}\psi_t\right)}{-2\psi_t \left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \qquad \text{A35}$$

$$\frac{\partial}{\partial_t} y_t = -\frac{\sqrt{1-\delta_t^2}\left(-\left(\frac{\partial}{\partial t} I_t\right) - 2\delta_t\left(\frac{\partial}{\partial t}\delta_t\right) + \delta_t^2\left(\frac{\partial}{\partial t} I_t\right)\right)c\left(\frac{\partial}{\partial t}\psi_t\right)}{-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \quad \text{A36}$$

Substituting the solutions for $\frac{\partial}{\partial t} y_t$ into A34 then yields two solutions for y_t. Similarly, Equations A37 and A38 differ only in sign.

$$y_t = -2\frac{(\delta_t^2 - 1)\sqrt{1-\delta_t^2}\,c\left(\frac{\partial}{\partial t}\psi_t\right)}{-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \quad \text{A37}$$

$$y_t = 2\frac{(\delta_t^2 - 1)\sqrt{1-\delta_t^2}\,c\left(\frac{\partial}{\partial t}\psi_t\right)}{-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \quad \text{A38}$$

Although the equation for range was added to the system of equations only for the purpose of transforming into normal polynomial form, the solution for range falls out easily from the Gröbner bases.

$$r_t = -2\frac{c\left(\frac{\partial}{\partial t}\psi_t\right)(\delta_t^2 - 1)}{-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)} \quad \text{A39}$$

Given a solution for range from Equation A39, it follows that the following solution exists for the intrinsic power of the sound source, derived from Equation 8 in Table 1.

$$\eta = 4\frac{c^2\left(\frac{\partial}{\partial t}\psi_t\right)^2(\delta_t^2 - 1)^2 e^{I_t}}{\left(-2\psi_t\left(\frac{\partial}{\partial t}\delta_t\right)^2 + \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\delta_t^2 - \left(\frac{\partial}{\partial t} I_t\right)\left(\frac{\partial}{\partial t}\psi_t\right)\right)^2} \quad \text{A40}$$